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The crossover between orthogonal and unitary symmetry in small disordered systems: a supersymmetry approach

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Abstract. We consider the spectral statistics of independent electrons moving at zero temperature in a weakly disordered metallic ring threaded by a magnetic flux. The analysis is based on the supersymmetry method involving both commuting and anticommuting variables. Besides, we consider an ensemble of Gaussian distributed symmetric random matrices (Gaussian orthogonal ensemble) which are perturbed by a small time reversal symmetry breaking contribution. For energies smaller than the inverse diffusion time around the ring E_c, the spectral correlation functions of both models can be represented in terms of supermatrix integrals of identical structure. In conformity with recent numerical results, this implies that the spectral properties of the two models coincide. These matrix integrals are to a large extent universal, i.e. they depend only on two physical parameters: the mean level spacing and a symmetry breaking parameter which is identified as the typical sensitivity of levels to the time reversal symmetry breaking perturbation. We parametrize the relevant matrix coset space of the nonlinear σ-model in a novel way which is particularly convenient for treating models in the crossover between the two symmetry classes. As an example, we present a detailed calculation of the level–level correlation function. The basic formalism, however, applies quite generally and can be used for the investigation of different types of correlation functions and system geometries as well.

1. Introduction and results

At low temperatures, small disordered metals, i.e. disordered metal probes of an extension less than or comparable with the temperature dependent phase coherence length, display various quantum effects which are related to the existence of strong correlations in their energy spectra. Typical examples of such phenomena are universal conductance fluctuations [1], or mesoscopic effects arising in the orbital magnetic susceptibility [2, 3]. Perhaps the most remarkable feature of these spectral correlations is their universality, i.e. the degree of correlation between two energy levels depends on their distance ω and a few macroscopic parameters characterizing the system but not the microscopic origin of stochasticity. Universal spectral fluctuations are shown by various other small stochastic quantum systems, such as complex nuclei, atoms or molecules [4] as well. Phenomenologically, they can very accurately be described by representing the Hamilton operators of the respective systems in terms of Gaussian distributed matrix Hamiltonians [4]. The spectral properties of such matrix ensembles are uniquely determined by two fundamental symmetries: Unitarity and time reversal symmetry. As long as spin-dependent interactions are inessential (the only case which will be considered in this paper), it is sufficient to distinguish between two

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cases: Time reversal invariant systems (orthogonal symmetry) described by ensembles of real symmetric matrices and systems of broken time reversal invariance (unitary symmetry) represented by Hermitian random matrices. The corresponding matrix ensembles are called the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE), respectively. Both symmetry classes display qualitatively different spectral properties.

The relevance of these concepts to the spectral theory of disordered metallic particles has been conjectured a long time ago by Gor'kov and Eliashberg [5]. Depending on the symmetries of the scattering potential, disordered metallic systems may belong to any one of the three universality classes: the unitary class, the orthogonal class or the symplectic class which is appropriate for time reversal invariant systems with broken central symmetry. Some years ago, it was shown [6] that random matrix theory is indeed an appropriate tool for describing the spectral statistics of isolated metal probes on energy scales smaller than the inverse diffusion time through the system E_c: Starting from a microscopic model Hamiltonian, the correlation functions describing the spectral properties of samples belonging to each symmetry class were calculated non-perturbatively. The analysis was based on the supersymmetry method involving both, commuting and anticommuting variables. In all three cases, the results turned out to coincide exactly with the correlation functions describing the corresponding random matrix ensemble. Correlations on scales larger than E_c fall beyond the scope of random matrix theory. In this regime, the spectral statistics is influenced by spatial fluctuations (cf section 2), whereas random matrix theory is only applicable to ergodic systems. In this paper, we will focus on the description of correlations on short scales $\omega < E_c$.

In the context of solid state physics, the orthogonal symmetry class corresponds to systems with pure potential impurity scattering while the unitary class is appropriate to cases where the time reversal invariance is broken, e.g. by a strong external magnetic field. At the same time, *weak* magnetic fields are powerful experimental tools for investigating electron interference phenomena like, e.g. weak localization effects, universal conductance fluctuations [7] or persistent currents in normal metallic rings [8, 9]. For that reason, it is of theoretical interest to consider not only the pure symmetry classes, but also the case of partly broken time reversal invariance, i.e. the field-driven crossover between orthogonal and unitary symmetry. Numerically, the spectral statistics of a disordered metallic ring pierced by a magnetic flux was recently investigated by Dupuis and Montambaux [10]. Their results clearly indicate, that random matrix theory is also applicable in the crossover region. Within the framework of random matrix theory, systems of intermediate symmetry are conveniently represented by Hamiltonians of the type

$$H = H_s + i\alpha H_a \qquad \alpha \in [0, 1] \tag{1.1}$$

where $H_s(H_a)$ are N-dimensional real symmetric (anti-symmetric) statistically independent random matrices of the same variance λ^2/N. The crossover is driven by the parameter α, $\alpha = 0(1)$ corresponding to the case of pure orthogonal (unitary) symmetry. For the first time, random matrix ensembles of this type have been investigated by Pandey and Mehta [11]. By applying a mathematicl formalism different from the supersymmetry method used in this paper, they were able to calculate the spectral k-level correlation functions in dependence on the parameter α without any approximation. In [10] the two-level correlation function obtained by Pandey and Mehta was found also to describe the spectral properties of the metallic system if the parameter α is related to the magnetic flux via

$$N\alpha^2 = 4\pi \frac{E_c}{\Delta} \left(\frac{\phi}{\phi_0}\right)^2 \tag{1.2}$$

where $E_c = \hbar\pi^2 D/L^2$ denotes the Thouless energy [12], D the diffusion constant, L the extension of the system, Δ the typical level spacing at the Fermi surface and $\phi_0 = 2\pi\hbar c/e$ the unit flux quantum.

Up to now all analytical results concerning the dependence of the level statistics on a weak magnetic field have exclusively been obtained by impurity perturbation theory, cf e.g. [1]. For small isolated particles and energy differences much larger than the mean level spacing, the perturbative results indeed agree with the predictions of random matrix. A direct comparison over the whole range of energies including the domain of strong level repulsion $\omega \simeq \Delta$, however, turns out to be impossible. As $\omega \to 0$, the results obtained in perturbation theory show an unphysical divergence which can only be removed by introducing an ad hoc cut-off parameter. On the other hand, it is just the region of level repulsion where the two symmetry classes differ most strongly from each other, i.e. close to the origin, the correlation functions of pure symmetry exhibit different power law behaviour and must be related to each other by a non-trivial crossover function. Hence, as far as the analysis of the symmetry crossover is concerned, the present stage of the theory with restriction to the regime $\omega \gg \Delta$ is unsatisfactory and motivates an extension of the non-perturbative analysis of [6] to the region of intermediate symmetry.

In this paper, we study the spectral fluctuations in the crossover region analytically within the formalism of the supersymmetric nonlinear σ-model [6]. The method will be applied to a system of independent electrons moving in a random potential (model 1) and to an ensemble of random Hamiltonians of the type (1.1). As for the random matrix case we distinguish between a randomly distributed perturbation H_a like in [11] (model 2) and a fixed antisymmetric realization of H_a (model 3). We find all model systems to be described by generating functionals $Z(s_i, \Delta_i)$ $(i = 1, 2, 3)$ of identical structure. These functionals are to a large extent universal, i.e. they depend only on two physical parameters, the mean level spacing Δ_i and a 'symmetry breaking parameter' s_i to be discussed below. Due to the equality of the generating functionals, the spectral properties of the model systems expressed in terms of the respective parameters Δ_i and s_i coincide quantitatively. This coincidence is not restricted to the two-point correlation functions considered in this paper. It is straightforward to show that up to an increase in the dimension of the matrix valued fields (Q in section 2), the generating functional appropriate for the calculation of higher order correlation functions has the same form as the one considered in this paper, i.e. the results obtained in the three different models again have to coincide. The increase in the number of integration variables, however, renders the concrete evaluation of these higher order functionals more and more complicated.

For the symmetry breaking parameters s_i we obtain

$$\begin{aligned} s_1 &= 4\pi \frac{E_c}{\Delta}\left(\frac{\phi}{\phi_0}\right)^2 && \text{model 1} \\ s_2 &= N\alpha^2 && \text{model 2} \\ s_3 &= \frac{\operatorname{tr}(H_a H_a^T)}{\lambda^2}\alpha^2 && \text{model 3.} \end{aligned} \tag{1.3}$$

The last form s_3 points to the physical meaning of the parameters s_i as measures of the typical sensitivity of energy levels to the time reversal symmetry breaking perturbation. As for model 1, it has been shown by Thouless [12], that a typical level at the Fermi surface changes by the order of $E_c(\phi/\phi_0)^2$ if the ring is threaded by a small flux ϕ. Hence, s_1 is proportional to the flux sensitivity of energy levels in the spectral domain under

consideration, measured in units of the mean level spacing. A consideration similar to Thouless's original one shows that this interpretation of the s_i holds in the random matrix case as well: According to second order perturbation theory, the energy E_n corresponding to an eigenstate $|n\rangle$ of the unperturbed Hamiltonian H_s changes by

$$\frac{\delta E_n}{\Delta} = \frac{\alpha^2}{\Delta} \sum_{m \neq n} \frac{|\langle n|H_a|m\rangle|^2}{E_n - E_m} + \mathcal{O}(\alpha^4) \tag{1.4}$$

under the influence of $i\alpha H_a$. Like in Thouless's derivation, we assume that the typical (i.e. n-averaged RMS) magnitude of the oscillating sum is determined by the contribution of adjacent levels $|E_n - E_m| \simeq \Delta$ and get

$$\begin{aligned}\left(\frac{\delta E_n}{\Delta}\right)_{\text{typ}} &\simeq \frac{\alpha^2}{\Delta^2}|\langle n|H_a|m\rangle|^2 \\ &= \frac{\alpha^2}{\Delta^2}(H_a)_{\mu\nu}(H_a^T)_{\nu'\mu'}\langle n \mid \mu\rangle\langle\mu'|n\rangle\langle\nu \mid m\rangle\langle m|\nu'\rangle. \end{aligned} \tag{1.5}$$

Upon averaging over the ensemble of the H_s, $\langle n \mid \mu\rangle\langle\mu'|n\rangle \to N^{-1}\delta_{\mu\mu'}$, hence

$$\left\langle \left(\frac{\delta E_n}{\Delta}\right)_{\text{typ}} \right\rangle_{H_s} \simeq \frac{\alpha^2}{N^2\Delta^2}\text{tr}(H_a H_a^T) = \frac{\alpha^2}{\pi^2\lambda^2}\text{tr}(H_a H_a^T) \tag{1.6}$$

where we have used the fact that the GOE level spacing at the centre of the Wigner semicircle $\Delta = \pi\lambda/N$. The bracket $\langle\rangle_{H_s}$ denotes the average over the random distribution of H_s. In the case of a fixed H_a, this expression is proportional to the parameter s_3. In model 2, however, where H_a is random, we obtain

$$\left\langle \left(\frac{\delta E_n}{\Delta}\right)_{\text{typ}} \right\rangle_{H_s, H_a} \simeq \frac{\alpha^2 N}{\pi^2} \tag{1.7}$$

i.e. up to a constant s_2.

The analysis of the crossover region necessitates a complete reparametrization of the saddle point manifold central to the evaluation of the nonlinear σ-model, i.e. the coset space $\simeq$ UOSP(2, 2/4)/UOSP(2/2) $\times$ UOSP(2/2) [13]. For that reason, the calculation of the two-level correlation function will be described in detail below. Moreover, we hope that a detailed presentation of the formalism is of interest for its own sake, as it sheds some light on the interplay of the different degrees of freedom in dependence on the symmetry and may demonstrate the flexibility of the supersymmetry method in handling problems of partly broken symmetry in general. The formalism is not restricted to the calculation of spectral correlations, but can also be applied to the non-perturbative analysis of various other properties of metallic particles in the regime of weak fields. Due to the presence of a magnetic field which is neither weak nor strong as far as the symmetries of the system are concerned, however, the corresponding calculations will in general be more difficult than in the cases of pure orthogonal or unitary symmetry, respectively.

The paper is organized as follows. In section 2 we represent the two-level energy correlation function of a metallic ring in terms of a generating functional. The derivation of an analogous expression for the random matrix ensemble is presented in appendix B. In section 3 we restrict ourselves to the consideration of the ergodic regime ($\omega < E_c$) and calculate the two-level correlation function. We have tried to present the formalism in such a way that it can straightforwardly be applied to the calculation of other two-point correlation functions as well. Technical details are deferred to the appendices C and D. We conclude in section 4. Appendix A provides a summary of notation and conventions used in this paper.

2. Derivation of the supermatrix functional

In this section, we derive the functional integral representation for the energy correlation function describing a mesoscopic ring threaded by a flux. The final result of this section, (2.23) can be obtained by a few straightforward modifications of more general expressions contained in [6]. For pedagogical reasons and in order to introduce some notations required in forthcoming sections, we nevertheless outline its derivation here. For all details we refer to [6]. Readers who are familiar with the formalism of the supersymmetric nonlinear σ-model are invited to skip this section and to turn directly to section 3.

Let us consider a weakly disordered quasi one-dimensional mesoscopic ring threaded by a static magnetic flux ϕ. The circumference L is assumed to be much less than the localization length and transverse dimensions are of the order of the elastic mean free path $l \ll L$. We describe a system of non-interacting electrons moving in the ring by the one-particle Hamiltonian

$$H = \frac{1}{2m}\hat{p}^2 + V \qquad \hat{p} = -i\hbar\nabla + \hbar\frac{2\pi}{L}\frac{\phi}{\phi_0}\hat{e}_x. \tag{2.1}$$

Here $\hat{e}_x$ is the unit vector in the tangential direction along the ring and V a random white-noise potential defined by the correlator

$$\langle V(x)\rangle_V = 0 \qquad \langle V(x)V(y)\rangle_V = \frac{\hbar}{2\pi\nu\tau}\delta(x-y) \tag{2.2}$$

where the angular brackets denote the ensemble average, ν is the average density of states (per spin and volume) at the Fermi energy E_F and $\tau \gg \hbar E_F^{-1}$ the elastic scattering time. Our aim is to calculate the connected part of the dimensionless density–density correlation function (the 2-level cluster function)

$$Y_2 = \Delta^2\langle\rho(E+\omega/2)\rho(E-\omega/2)\rangle_V - 1 \tag{2.3}$$

where $\Delta = (\nu\mathcal{V})^{-1}$ denotes the mean level spacing at the Fermi energy, $\mathcal{V}$ is the system's volume and $\rho = \mathrm{tr}(\delta(E-H))$. The function Y_2 can be expressed by the product of an advanced and a retarded Green function according to

$$Y_2 = \frac{\Delta^2}{2\pi^2}\mathrm{Re}(\langle F(\omega)\rangle_V) - \frac{1}{2} \tag{2.4}$$

where

$$F(\omega) = \mathrm{tr}\left(\frac{1}{E-\omega^+/2-H}\right)\mathrm{tr}\left(\frac{1}{E+\omega^+/2-H}\right) \tag{2.5}$$

and $\omega^+ = \omega + i\delta$, δ positive infinitesimal. As long as $\omega \ll E \simeq E_F$, the function F does not explicitly depend on E [1,6].

Starting point of the supersymmetry approach is a representation of the Green functions in terms of Gaussian type integrals. Introducing a field of supervectors

$$\psi(x)^T = (S_1(x), \chi_1(x), S_2(x), \chi_2(x)) \tag{2.6}$$

where $S(\chi)$ are complex commuting (anti-commuting) variables, we obtain [6]

$$F(\omega) = \frac{1}{16}\left\langle \int \mathrm{d}x\, \psi^\dagger(x)(1+\Lambda)(1+L_g)\psi(x) \int \mathrm{d}y\, \psi^\dagger(y)(1-\Lambda)(1+L_g)\psi(y)\right\rangle_\psi$$
$$= \left\langle \int \mathrm{d}x\, \psi^\dagger(x) P(a,\mathrm{B})\psi(x) \int \mathrm{d}y\, \psi^\dagger(y) P(r,\mathrm{B})\psi(y)\right\rangle_\psi \qquad (2.7)$$

where

$$\langle \ldots\rangle_\psi = \int \mathrm{d}[\psi(x)]\mathrm{e}^{-L[\psi,\psi^\dagger]}(\ldots)$$
$$L[\psi,\psi^\dagger] = \mathrm{i}\int \mathrm{d}x\, \psi^\dagger(x)\Lambda^{1/2}\left[E - \frac{\omega^+}{2}\Lambda - H(x)\right]\Lambda^{1/2}\psi(x) \qquad (2.8)$$
$$P(a,\mathrm{B}) = \tfrac{1}{4}(1+\Lambda)(1+L_g) \qquad P(r,\mathrm{B}) = \tfrac{1}{4}(1-\Lambda)(1+L_g)$$

with $\mathrm{d}[\psi] = \prod_{p=1,2} 2\,\mathrm{d}\mathrm{Re}\,S_p\,\mathrm{d}\mathrm{Im}\,S_p\,\mathrm{d}\chi_p^*\,\mathrm{d}\chi_p$, $L_g = \mathrm{diag}(1,-1,1,-1)$ and $\Lambda = \mathrm{diag}(1,1,-1,-1)$. Note that the matrix k in [6] is equal to $-L_g$. General definitions of the matrices L_g, Λ and the projection operators P are given in appendix A. The kernel $E-H$ is to be interpreted as a multiple of the four-dimensional unit matrix.

In order to arrive at a manageable expression after ensemble-averaging, we manipulate the action of the functional (2.8) according to

$$L[\psi,\psi^\dagger] = \tfrac{1}{2}(L[\psi,\psi^\dagger] + L[\psi,\psi^\dagger]^T)$$
$$= \frac{\mathrm{i}}{2}\int \mathrm{d}x\, \Psi^\dagger(x)\Lambda^{1/2}\left[E - \frac{\omega^+}{2}\Lambda - \frac{\hat{P}^2}{2m} - V(x)\right]\Lambda^{1/2}\Psi(x)$$
$$\Psi = \begin{pmatrix}\Psi_{d=1}\\ \Psi_{d=2}\end{pmatrix} := \begin{pmatrix}\psi\\ \psi^*\end{pmatrix} \qquad \frac{\hat{P}}{\hbar} = \frac{1}{\mathrm{i}}\nabla - \frac{2\pi}{L}\frac{\phi}{\phi_0}\tau_3 \qquad \tau_3 = \{(-1)^d\delta_{dd'}\}. \qquad (2.9)$$

For future reference, we have defined a set of additional indices $d = 1,2$ (GOE-indices) by $\Psi_{d=1} = \psi$, $\Psi_{d=2} = \psi^*$. As it stands, this transformation may seem to be quite artificial. Heuristically, however, ψ may be regarded as a sort of a wavefunction, whence the transformation $\psi \to \psi^*$, $H \to H^T$ becomes the analogue of the quantum mechanical time reversal transformation [4]. Still, the necessity of introducing the vector of doubled dimensionality, Ψ, can be made more apparent within the random matrix model discussed in appendix B (cf e.g. (B9), see also [14, 15]). The newly defined vector Ψ fulfils the 'time reversal' symmetry relation

$$\Psi^* = C\Lambda\Psi \qquad (2.10)$$

with an orthogonal matrix C given explicitly in appendix A. The disorder average can now easily be carried out and we arrive at

$$\langle \ldots\rangle_V = \int \mathrm{d}[\psi]\mathrm{e}^{-\mathcal{L}[\Psi]}(\ldots),$$
$$\mathcal{L}[\Psi] = \frac{\mathrm{i}}{2}\int \mathrm{d}x\, \Psi^\dagger(x)\Lambda^{1/2}\left[E - \frac{\omega^+}{2}\Lambda - \frac{\hat{P}^2}{2m}\right]\Lambda^{1/2}\Psi(x)$$
$$+ \frac{\hbar}{16\pi\nu\tau}\int \mathrm{d}x\, (\Psi^\dagger(x)\Lambda\Psi(x))^2. \qquad (2.11)$$

The next step in the derivation of the nonlinear σ-model is a decoupling of those contributions to the quartic term Ψ^4 which vary slowly in comparison with the Fermi wavelength by means of a Hubbard–Stratonovich transformation. As a result the Ψ-integration becomes Gaussian and we obtain [6]

$$\langle F(\omega)\rangle_V = -\tfrac{1}{4}\int \mathrm{d}[Q(x)]\mathrm{e}^{-F[Q]}\int \mathrm{d}x\,\mathrm{STr}[g(x,x)P(a,\mathrm{B})]\int \mathrm{d}y\,\mathrm{STr}[g(y,y)P(r,\mathrm{B})]$$

$$F[Q] = \int \mathrm{d}x\,\mathrm{STr}\left[\frac{\pi\hbar\nu}{8\tau}Q(x)^2 - \tfrac{1}{2}(\ln g^{-1})(x,x)\right]$$

$$g^{-1} = \mathrm{i}\left(E - \frac{\omega^+}{2}\Lambda - \frac{\hat{P}^2}{2m} - \frac{\mathrm{i}\hbar}{2\tau}Q\right) \tag{2.12}$$

where Q is a slowly varying field of eight-dimensional supermatrices having a similar algebraic structure as the dyadic product $\Lambda^{1/2}\Psi\Psi^\dagger\Lambda^{1/2}$, i.e.

$$Q^\dagger = KQK \qquad Q^T = C^TQC. \tag{2.13}$$

Definitions of the matrix K and the supertrace 'STr' can be found in appendix A. The symbol $\int \mathrm{d}[Q]$ denotes the integration over all independent components of the matrix Q. We evaluate the functional integral (2.12) further by means of a saddle point approximation. The spatially constant saddle-point matrices Q_{sp} are determined by the stationarity condition

$$\frac{\delta}{\delta Q_{\mathrm{sp}}}F[Q]\bigg|_{Q=Q_{\mathrm{sp}}} = 0. \tag{2.14}$$

The solutions of this equation form the saddle point manifold and can be represented as

$$Q_{\mathrm{sp}} = \hat{T}^{-1}\Lambda\hat{T} \tag{2.15}$$

where the matrices $\hat{T}$ are defined by symmetry relations resulting from (2.13):

$$\hat{T}^\dagger = K\hat{T}^{-1}K \qquad \hat{T}^T = C^T\hat{T}^{-1}C. \tag{2.16}$$

Since we are concerned with a quasi-one-dimensional ring, the slowly varying fields $Q(x)$ do not significantly fluctuate in transverse direction, i.e. there is no diffusion in transverse direction. Allowing for small spatial fluctuations in longitudinal direction $Q(x) = \hat{T}^{-1}(x)\Lambda\hat{T}(x)$ around a given constant saddle point Q_{sp} and expanding to lowest order in the small parameters l/L and $\omega\tau$, we obtain the functional

$$\langle\ldots\rangle_Q = \int [\mathrm{d}Q(x)]\mathrm{e}^{-F[Q]}(\ldots)$$

$$F[Q] = \frac{\pi\nu S}{8}\int \mathrm{d}x\,\mathrm{STr}[\hbar D(D_xQ(x))^2 + 2\mathrm{i}\omega^+\Lambda Q(x)]$$

$$D_x = \partial_x - \mathrm{i}\frac{2\pi}{L}\frac{\phi}{\phi_0}[\tau_3,\ ] \tag{2.17}$$

where D denotes the diffusion constant, S the cross-sectional area of the ring and $[dQ]$ is the volume element of the saddle point manifold. Note that the functional (2.17) is invariant under the gauge transformation

$$Q(x) \to \exp\left(-i\tau_3 \frac{2\pi}{L}\frac{n}{2}x\right) Q(x) \exp\left(i\tau_3 \frac{2\pi}{L}\frac{n}{2}x\right) \qquad n \in \mathcal{Z}. \tag{2.18}$$

Since this transformation is equivalent to $\phi \to \phi + n\phi_0/2$, our expression is periodic in $\phi_0/2$ [16]. It can also easily be shown that Y_2 is even in the flux, hence we may restrict ourselves to flux values $\phi \in [0, \phi_0/4]$ in all further considerations.

In order to elucidate the notion of a quasi-zero-dimensional, ergodic system brought up in the introduction, let us briefly outline how the generating functional (2.17) may be analysed perturbatively. At the same time, this will reveal the correspondence between fluctuations of the field $\hat{T}$ on the one hand and the diffusion modes playing a central role in the diagrammatic approach on the other hand. One possible parametrization of the matrices $\hat{T}$ is given by (C1). The two-dimensional supermatrices $t_{pp'}^{dd'}$ appearing in (C1) are the basic degrees of freedom underlying the perturbative evaluation of the generating functional. Inserting that parametrization into $F[Q]$, we obtain in a symbolic notation

$$F[Q] = \sum_{m=1}^{\infty} F^{(2m)}[t] \tag{2.19}$$

where $F^{(2m)}[t]$ is an abbreviation for a term of order $2m$ in the t-fields. Explicitly, the free term $F^{(2)}[t]$ takes the form

$$F^{(2)}[t] = -\frac{2\pi}{\Delta}\sum_n (\Pi_D^{-1}(p_n)\mathrm{STr}(t_{21}^{11}(-p_n)t_{12}^{11}(p_n)) + \Pi_C^{-1}(p_n)\mathrm{STr}(t_{21}^{21}(-p_n)t_{12}^{12}(p_n))) \tag{2.20}$$

where

$$\begin{aligned} \Pi_D^{-1}(p_n) &= \hbar D p_n^2 - i\omega^+ \\ \Pi_C^{-1}(p_n) &= \hbar D\left(p_n + 2\frac{2\pi}{L}\frac{\phi}{\phi_0}\right)^2 - i\omega^+ \\ p_n &= 2\pi n/L. \end{aligned} \tag{2.21}$$

In other words, the free part of the action corresponding to the elementary t-fields contained in $\hat{T}_D$ ($\hat{T}_C$) is just the inverse of the diffuson (cooperon)-propagator as it appears e.g. in [1]. The functional can now be evaluated perturbatively by expanding the weighting factor $\exp(-F)$ around the free part in powers of t-fields and performing the integration by means of Wick's theorem. This procedure results in the complete series of interacting diffusion modes as it can alternatively be obtained by the standard impurity perturbation method. Details of a perturbative analysis with emphasis on the effects caused by a weak field can be found in [14, 15].

As a consequence of the isolated ring geometry, the n-summation in $F^{(2)}[t]$ extends over all integers, *including zero*. For

$$\omega \ll \frac{\hbar\pi^2 D}{L^2} = E_c \qquad \frac{\phi}{\phi_0} \ll 1 \tag{2.22}$$

the contribution of higher modes to the weighting factor $\exp(-F^{(2)}[t])$ is exponentially small in comparison with that of the zero-mode. In this regime, spatial fluctuations corresponding to $n \neq 0$ become inessential and the system can be regarded as effectively pointlike. For $\omega, \phi \to 0$, the propagators $\Pi_{j=C,D}(p_n = 0)$ eventually diverge. This infrared divergence of the diagrammatic method necessitates a non-perturbative analysis as it is presented in the next section.

Henceforth, we will restrict ourselves to the consideration of the 'non-perturbative' regime defined by (2.22). Retaining only the zero-mode contribution and replacing $Q(x)$ by the spatially constant matrix Q in (2.17), we arrive at the final result of this section

$$\langle F(\omega)\rangle_V = \left(\frac{\pi}{\Delta}\right)^2 \left\{1 - \tfrac{1}{4}\int [\mathrm{d}Q]\mathrm{e}^{-F[Q]}\mathrm{STr}[P(a,\mathrm{B})(Q-\Lambda)]\mathrm{STr}[P(r,\mathrm{B})(Q-\Lambda)]\right\}$$

$$F[Q] = F_0[Q] + F_1[Q] \tag{2.23}$$

$$F_0[Q] = \mathrm{i}\frac{\pi\omega^+}{4\Delta}\mathrm{STr}(Q\Lambda) \qquad F_1[Q] = -\frac{s_1}{8}\mathrm{STr}([Q,\tau_3])^2 \qquad s_1 = 4\pi\frac{E_c}{\Delta}\left(\frac{\phi}{\phi_0}\right)^2$$

where s_1 is the symmetry breaking parameter defined in (1.3) and Q denotes now a single Q-matrix specified by the symmetry relations (2.13).

It is shown in appendix B that the 'free energy' $F[Q]$ of the Q-integral describing a single N-dimensional random Hamiltonian of partly broken orthogonal symmetry, $H = H_s + \mathrm{i}\alpha H_a$, $\alpha \ll 1$, is given by

$$F[Q] = \mathrm{i}\frac{\pi\omega^+}{4\Delta}\mathrm{STr}(Q\Lambda) - \frac{N\alpha^2}{8}\mathrm{STr}([Q,\tau_3]^2)$$

which means that the model of independent electrons moving in a random potential and the random matrix model exhibit identical statistical properties if we identify the respective parameters according to (1.2).

3. Calculation of the two-level correlation function

We now turn to the concrete calculation of the probability $R(x) = Y_2(x) + 1 - \pi\delta(x)$ for finding two energy levels a distance $\omega = x\Delta/\pi$ apart from each other. Expressed in terms of the function $F(\omega)$ defined in the previous section, $R(x)$ reads

$$R(x) = \frac{\Delta^2}{2\pi^2}\mathrm{Re}(\langle F(\omega)\rangle_V) + \tfrac{1}{2} - \pi\delta(x) \tag{3.1}$$

where the integral representation for $\langle F(\omega)\rangle_V$ is given by equation (2.23) with

$$Q = \hat{T}^{-1}\Lambda\hat{T}. \tag{3.2}$$

In order to perform the integration in (2.23), the matrices $\hat{T}$ have to be specified according to some concrete parametrization. In our case, the presence of the symmetry breaking term $\mathrm{STr}([Q,\tau_3]^2)$ suggests to start from the ansatz

$$\hat{T} = \hat{T}_C\hat{T}_D \tag{3.3}$$

where both $\hat{T}_C$ and $\hat{T}_D$ individually fulfil the symmetry relations (2.16) but additionally

$$[\hat{T}_D, \tau_3] = 0 \tag{3.4}$$

such that only degrees of freedom associated with the matrix $\hat{T}_C$ appear in $\mathrm{STr}([Q, \tau_3]^2)$. Standard parametrizations [6, 13] which are not based on the decomposition (3.3) lead to a nonlinear mixing between degrees of freedom with different GOE-index $d = 1, 2$. As a consequence, the symmetry breaking term takes a very complicated form involving almost all independent components of the Q-matrices [17]. Below we specify a concrete parametrization of the Q matrices based on the decomposition (3.3) and apply it to calculating the density–density correlation function. The derivation of this parametrization as well as the calculation of the associated volume element is deferred to appendices C and D.

Similarly to [6], we represent the matrices $\hat{T}(j)$ in a quasi-diagonalized form

$$\hat{T}_j = [V_2(j)V_1(j)]^{-1}\hat{T}_j^0 V_2(j)V_1(j) \qquad j = \mathrm{C}, \mathrm{D} \tag{3.5}$$

with 'eigenvalue' matrices $\hat{T}_j^0$ given by

$$\begin{aligned}
\hat{T}_D^0 &= \begin{pmatrix} \cos(\hat{\theta}(\mathrm{D})/2) & \mathrm{i}\sin(\hat{\theta}(\mathrm{D})/2) \\ \mathrm{i}\sin(\hat{\theta}(\mathrm{D})/2) & \cos(\hat{\theta}(\mathrm{D})/2) \end{pmatrix} \\
\hat{T}_C^0 &= \begin{pmatrix} \cos(\hat{\theta}(\mathrm{C})/2) & \mathrm{i}\sin(\hat{\theta}(\mathrm{C})/2) \times C_0 \\ \mathrm{i}\sin(\hat{\theta}(\mathrm{C})/2) \times C_0^T & \cos(\hat{\theta}(\mathrm{C})/2) \end{pmatrix} \\
\hat{\theta}(j) &= \begin{pmatrix} \mathrm{i}\theta_B(j) & \\ & \theta_F(j) \end{pmatrix} \qquad \theta_B(j), \theta_F(j) \in \mathcal{R},\ j = \mathrm{C}, \mathrm{D}.
\end{aligned} \tag{3.6}$$

The appropriate range of all integration variables like e.g. $\theta_B(j)$ and $\theta_F(j)$ will be determined in appendix C and can be read off from (3.16). The matrix C_0 is defined in appendix A. The 'diagonalizing' matrices $V_1(j)$ and $V_2(j)$ are respectively defined by

$$V_1(j) = \mathrm{diag}((v_1(j)^T)^{-1}, v_1(j), (v_2(j)^T)^{-1}, v_2(j)) \tag{3.7}$$

with

$$v_1(j) = \exp\begin{pmatrix} & -\xi_1^*(j) \\ \xi_1(j) & \end{pmatrix} \qquad v_2(j) = \exp \mathrm{i}\begin{pmatrix} & -\xi_2^*(j) \\ \xi_2(j) & \end{pmatrix} \tag{3.8}$$

and

$$\begin{aligned}
V_2(j) &= \begin{pmatrix} \exp(\mathrm{i}\hat{\phi}(j) \times \tau_3) & \\ & 1\!\!1_4 \end{pmatrix} \\
\hat{\phi}(j) &= \begin{pmatrix} \phi_B(j) & \\ & \phi_F(j) \end{pmatrix} \qquad \phi_B(j), \phi_F(j) \in \mathcal{R}.
\end{aligned} \tag{3.9}$$

The variables $\xi_p(j)$ are anticommuting. Combining (3.2), (3.3) and (3.5), we obtain

$$\begin{aligned}
Q &= U_D^{-1} V_C^{-1} Q_C^0 V_C U_D \\
V_C &= V_2(\mathrm{C})V_1(\mathrm{C}) \\
U_D &= V_2(\mathrm{D})^{-1}\hat{T}_D^0 V_2(\mathrm{D})V_1(\mathrm{D})
\end{aligned} \tag{3.10}$$

with

$$Q_{\mathrm{C}}^{0}=\begin{pmatrix}\cos(\hat{\theta}(\mathrm{C})) & \mathrm{i}\sin(\hat{\theta}(\mathrm{C}))\times C_0\\ -\mathrm{i}\sin(\hat{\theta}(\mathrm{C}))\times C_0^{T} & -\cos(\hat{\theta}(\mathrm{C}))\end{pmatrix}. \tag{3.11}$$

It turns out to be more convenient to use

$$\lambda_{\mathrm{B}}(j)=\cosh(\theta_{\mathrm{B}}(j)) \qquad \lambda_{\mathrm{F}}(j)=\cos(\theta_{\mathrm{F}}(j)) \tag{3.12}$$

instead of $\theta_g(j)$ as independent integration variables. In terms of these conventionally named 'eigenvalues', the symmetry-breaking term takes the simple form

$$F_1[Q]=2s_1(\lambda_{\mathrm{B}}(\mathrm{C})^2-\lambda_{\mathrm{F}}(\mathrm{C})^2) \tag{3.13}$$

depending exclusively on C-eigenvalues. The energy-dependent term is given by

$$\begin{aligned}F_0[Q]=-\mathrm{i}x[&(\lambda_{\mathrm{B}}(\mathrm{C})\lambda_{\mathrm{B}}(\mathrm{D})-\lambda_{\mathrm{F}}(\mathrm{C})\lambda_{\mathrm{F}}(\mathrm{D}))+(\lambda_{\mathrm{B}}(\mathrm{C})-\lambda_{\mathrm{F}}(\mathrm{C}))(\lambda_{\mathrm{B}}(\mathrm{D})\\ &-\lambda_{\mathrm{F}}(\mathrm{D}))(\xi_2^{*}(\mathrm{C})\xi_2(\mathrm{C})-\xi_1^{*}(\mathrm{C})\xi_1(\mathrm{C}))/2].\end{aligned} \tag{3.14}$$

It is not difficult to verify that the Q-matrices parametrized by (3.10) fulfil the symmetry relations (2.13). Still, however, the integration regions of the independent variables have to be specified in such a way that the newly parametrized Q-matrices faithfully represent the whole saddle-point manifold. This is done in appendix C by comparing (3.10) with the parametrization used in model I of [6]. Finally, the volume element of the saddle point manifold associated with the new parametrization has to be calculated (cf appendix D). As a result of all these we obtain

$$\int[\mathrm{d}Q]=\int \mathrm{d}\mu(\mathrm{C})\int \mathrm{d}\mu(\mathrm{D})\left(\frac{2\lambda_{\mathrm{F}}(\mathrm{C})}{\lambda_{\mathrm{B}}(\mathrm{C})+\lambda_{\mathrm{F}}(\mathrm{C})}\right)^{2} \tag{3.15}$$

where

$$\begin{aligned}&\int \mathrm{d}\mu(j)=(2\pi)^2\int \mathrm{d}\tilde{\mu}(j)\int \mathrm{d}\xi_1(j)\,\mathrm{d}\xi_2(j)\,\mathrm{d}\xi_1^{*}(j)\,\mathrm{d}\xi_2^{*}(j)\\ &\int \mathrm{d}\tilde{\mu}(j)=\int_0^{2\pi}\frac{\mathrm{d}\phi_{\mathrm{B}}(j)}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi_{\mathrm{F}}(j)}{2\pi}\int_1^{\infty}\mathrm{d}\lambda_{\mathrm{B}}(j)\int_{b(j)}^{1}\mathrm{d}\lambda_{\mathrm{F}}(j)\frac{1}{(\lambda_{\mathrm{B}}(j)-\lambda_{\mathrm{F}}(j))^2}\end{aligned} \tag{3.16}$$

and $b(j)=0(-1)$ for $j=\mathrm{C}(\mathrm{D})$. Apart from a halved integration interval of the C-fermion eigenvalue $\lambda_{\mathrm{F}}(\mathrm{C})$, both $\mathrm{d}\mu(\mathrm{C})$ and $\mathrm{d}\mu(\mathrm{D})$ have the form of GUE-volume elements (cf [6]).

The measure terms $\mathrm{d}\mu(j)$ diverge at $\lambda_{\mathrm{B}}(j)=\lambda_{\mathrm{F}}(j)=1$. As a consequence [18] one must not directly apply formula (A9) for the integration over anticommuting variables, i.e., terms of lower order in the anticommuting variables carefully have to be taken into account. A particular pedagogical discussion of these divergencies and their effect on the integration over Q-matrices of unitary symmetry can be found in [19]. Here, we are integrating twice over measures of GUE-type and straightforward extension of the results of [19] leads to the following prescription for the integration procedure: As a first step, the integrand, which we temporarily denote by f, has to be expanded in powers of anticommuting variables according to

$$f=f_{00}+f_{04}\xi^4(\mathrm{D})+f_{40}\xi^4(\mathrm{C})+f_{44}\xi^8+\cdots \tag{3.17}$$

where f_{00} etc are functions of commuting variables and we introduced the abbreviations $\xi_1(j)\xi_2(j)\xi_1^*(j)\xi_2^*(j) =: \xi^4(j)$ and $\xi^4(\mathrm{C})\xi^4(\mathrm{D}) =: \xi^8$ for the sake of notational compactness. As was shown in [19], only expansion coefficients $f_{i_\mathrm{C} i_\mathrm{D}}$, $i_\mathrm{C}, i_\mathrm{D} = 0, 4$ lead to non-vanishing contributions after integration. A direct application of the integral theorem in [19] then leads to the result

$$\int [\mathrm{d}Q] f = h_{00} + h_{04} + h_{40} + h_{44} \tag{3.18}$$

where

$$\begin{aligned} h_{00} &= f_{00}(x_\mathrm{C} = 0, x_\mathrm{D} = 0) \\ h_{04} &= \int \mathrm{d}\tilde{\mu}\,(\mathrm{D})\, f_{04}(x_\mathrm{C} = 0, x_\mathrm{D}) \\ h_{40} &= \int \mathrm{d}\tilde{\mu}\,(\mathrm{C}) \left(\frac{2\lambda_\mathrm{F}(\mathrm{C})}{\lambda_\mathrm{B}(\mathrm{C}) + \lambda_\mathrm{F}(\mathrm{C})}\right)^2 f_{40}(x_\mathrm{C}, x_\mathrm{D} = 0) \\ h_{44} &= \int \mathrm{d}\tilde{\mu}\,(\mathrm{C}) \left(\frac{2\lambda_\mathrm{F}(\mathrm{C})}{\lambda_\mathrm{B}(\mathrm{C}) + \lambda_\mathrm{F}(\mathrm{C})}\right)^2 \int \mathrm{d}\tilde{\mu}\,(\mathrm{D})\, f_{44}(x_\mathrm{C}, x_\mathrm{D}). \end{aligned} \tag{3.19}$$

Here, $x_{j=\mathrm{C},\,\mathrm{D}}$ is an abbreviation for all commuting variables, i.e., $x_j = \{\theta_g(j), \phi_g(j); g = \mathrm{B}, \mathrm{F}\}$. Due to the damping factor (3.13), the terms h_{40} and h_{44} vanish in the case of strong symmetry-breaking fields $s_1 \gg 1$, i.e.

$$\phi \gg \phi_\mathrm{c} := \phi_0(\Delta/E_\mathrm{c})^{1/2} \tag{3.20}$$

and (3.18) reduces to the standard GUE-expression [19].

Let us now apply these general considerations to the integrand appearing in (2.23). Insertion of the parametrization (3.10) into

$$g = -\tfrac{1}{4}\mathrm{STr}[P(a, \mathrm{B})(Q - \Lambda)]\,\mathrm{STr}[P(r, \mathrm{B})(Q - \Lambda)] \tag{3.21}$$

leads to

$$\begin{aligned} g = {} & (1 - \lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}))^2 + \xi^4(\mathrm{C})(\lambda_\mathrm{B}(\mathrm{C}) - \lambda_\mathrm{F}(\mathrm{C}))^2(\lambda_\mathrm{B}(\mathrm{D})^2 + 1)/2 \\ & + \xi^4(\mathrm{D})(\lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{C})\lambda_\mathrm{F}(\mathrm{D}))^2 - \xi^4(\mathrm{D})(\xi_1^*(\mathrm{C})\xi_1(\mathrm{C}) - \xi_2^*(\mathrm{C})\xi_2(\mathrm{C})) \\ & \times (\lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{C})\lambda_\mathrm{F}(\mathrm{D}))(\lambda_\mathrm{B}(\mathrm{C}) - \lambda_\mathrm{F}(\mathrm{C}))(\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{D})) \\ & + \xi^8(\lambda_\mathrm{B}(\mathrm{C}) - \lambda_\mathrm{F}(\mathrm{C}))^2(\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{D}))^2/2 + (\text{non-contributing terms}). \end{aligned} \tag{3.22}$$

The complete integrand is given by $f = g\exp(-F_0[Q] - F_1[Q])$. Combining (3.13), (3.14) and (3.22), we arrive at

$$\begin{aligned} f_{00} &= (1 - \lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}))^2 g_0 \\ f_{04} &= (\lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{C})\lambda_\mathrm{F}(\mathrm{D}))^2 g_0 \\ f_{40} &= \tfrac{1}{2}(\lambda_\mathrm{B}(\mathrm{C}) - \lambda_\mathrm{F}(\mathrm{C}))^2(\lambda_\mathrm{B}(\mathrm{D})^2 + 1) g_0 \\ f_{44} &= \tfrac{1}{2}(\lambda_\mathrm{B}(\mathrm{C}) - \lambda_\mathrm{F}(\mathrm{C}))^2(\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{D}))^2 g_0 \\ & \quad \times \left(1 + 2ix(\lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{C})\lambda_\mathrm{F}(\mathrm{D})) - \frac{x^2}{2}(\lambda_\mathrm{B}(\mathrm{C})\lambda_\mathrm{B}(\mathrm{D}) - \lambda_\mathrm{F}(\mathrm{C})\lambda_\mathrm{F}(\mathrm{D}))^2\right) \end{aligned} \tag{3.23}$$

with

$$g_0 = \exp[ix(\lambda_B(C)\lambda_B(D) - \lambda_F(C)\lambda_F(D)) - 2s_1(\lambda_B(C)^2 - \lambda_F(C)^2)]. \tag{3.24}$$

The integrations over angle variables, $\phi_g(j)$, and D-eigenvalues can easily be carried out and we obtain

$$\begin{aligned}
h_{00} &= 0 \\
h_{04} &= 2i\sin(x)\exp(ix)/x^2 \\
h_{40} &= 4\int_1^\infty d\lambda_B(C)\int_0^1 d\lambda_F(C)\left[\frac{\lambda_F(C)}{\lambda_B(C)+\lambda_F(C)}\right]^2 \\
&\quad \times \exp[ix(\lambda_B(C)-\lambda_F(C)) - 2s_1(\lambda_B(C)^2-\lambda_F(C)^2)] \\
h_{44} &= \left(\frac{d}{dx}\right)^2 x^2 \int_1^\infty d\lambda_B(C)\int_0^1 d\lambda_F(C)\int_1^\infty d\lambda_B(D)\int_{-1}^1 d\lambda_F(D)\left[\frac{\lambda_F(C)}{\lambda_B(C)+\lambda_F(C)}\right]^2 g_0 \\
&= 2i\int_1^\infty d\lambda_B(C)\int_0^1 d\lambda_F(C)\frac{\lambda_F(C)}{\lambda_B(C)(\lambda_B(C)+\lambda_F(C))^2} \\
&\quad \times \exp[ix\lambda_B(C) - 2s_1(\lambda_B(C)^2-\lambda_F(C)^2)] \\
&\quad \times [-(\lambda_B(C)^2+\lambda_F(C)^2)\sin(x\lambda_F(C)) + 2i\lambda_B(C)\lambda_F(C)\cos(x\lambda_F(C))].
\end{aligned} \tag{3.25}$$

Adding these terms leads to

$$\begin{aligned}
\langle g\rangle_Q &= 2i\exp(ix)\sin(x)/x^2 - 2i\int_1^\infty d\lambda_B(C)\frac{\exp(ix\lambda_B(C))}{\lambda_B(C)}e^{-2s_1\lambda_B(C)^2} \\
&\quad \times \int_0^1 d\lambda_F(C)\lambda_F(C)\sin(x\lambda_F(C))e^{2s_1\lambda_F(C)^2}.
\end{aligned} \tag{3.26}$$

In calculating the real part of this expression, the positive infinitesimal imaginary part of the variable x has to be taken into account. In the vicinity of the origin, $\langle g\rangle_Q \sim 2i/x$, i.e. $\mathrm{Re}\langle g\rangle_Q \sim i(1/(x+i\eta) - 1/(x-i\eta)) = 2\pi\delta(x)$. As the final result of this section, we obtain

$$\begin{aligned}
R(x) &= 1 - \left(\frac{\sin(x)}{x}\right)^2 + \int_1^\infty d\lambda_B(C)\frac{\sin(x\lambda_B(C))}{\lambda_B(C)}e^{-2s_1\lambda_B(C)^2} \\
&\quad \times \int_0^1 d\lambda_F(C)\lambda_F(C)\sin(x\lambda_F(C))e^{2s_1\lambda_F(C)^2}.
\end{aligned} \tag{3.27}$$

In the absence of a symmetry-breaking field, $s_1 = 0$, the $\lambda_F(C)$-integration can be carried out and (3.27) reduces to the GOE-expression (5.42) of [6]. Even in this case, however, the present parametrization enables one to calculate $R(x)$ more directly. When using the parametrization in [6] one has to proceed indirectly by first calculating the x-Fourier transform of $R(x)$ [6].

4. Conclusion

In this paper we investigated two kinds of statistical models in a region of partly broken time reversal invariance: a system of independent electrons moving in a mesoscopic ring threaded by a flux on the one hand and an ensemble of real symmetric random Hamiltonians which are perturbed by a small antisymmetric contribution on the other hand. Our main concern was the description of the crossover between pure orthogonal and unitary symmetry, respectively. The analysis was based on the formalism of the supersymmetric nonlinear σ-model.

It turned out that both models are described by generating functionals of identical structure which means that their statistical properties coincide provided the respective physical parameters are identified appropriately. As a quantity of main interest, we calculated the two-level correlation function. Our result (3.27) has been obtained earlier within the framework of random matrix theory by means of an entirely different mathematical formalism [11]. Let us therefore, from our subjective point of view, comment on some characteristics of the supersymmetry method. We have derived a novel parametrization of the nonlinear σ-model which is particularly convenient for problems with emphasis on the crossover between two symmetry classes. This parametrization allows for a calculation of various statistical observables relating two energy levels to each other. Besides the two-level correlation function typical examples are current–level [17] or current–current [20] correlation functions. These functions can be obtained by manipulating the preexponential terms in the generating functional which does not lead to a significant change in the computational efforts needed to calculate the integral. In this aspect, the nonlinear σ-model is more suitable for applications in condensed matter physics than random matrix theory where observables like e.g. a current are not even defined. Moreover, the Q-matrices can be related to the diffusion modes describing many properties of weakly disordered metals. As is most obvious from a perturbative analysis, there is a one-to-one correspondence between spatial fluctuations of the fields $\hat{T}_C$ and $\hat{T}_D$ on the one hand and cooperon and diffuson on the other hand. By adding to the zero mode, as it was treated in this paper, the perturbatively evaluated contribution of higher modes, one may investigate *local* correlation functions, like e.g. fluctuations in the local electron density.

On the other hand, an investigation of correlation functions between more than two levels necessitates an enlargement of the Q-matrices and it is not clear to us whether the resulting integrals can still be worked out. In this aspect, the nonlinear σ-model is inferior to the standard methods employed in random matrix theory.

Especially in connection with mesoscopic rings in magnetic fields the role played by spin-orbit interactions has attracted much recent interest [2,21]. An application of the present formalism to the symplectic-unitary crossover will be the subject of a forthcoming publication.

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Appendix A. Notation

Throughout this paper, we frequently refer to results which were derived in [6] and [13]. Unfortunately, the notation and conventions used in these references differ considerably. In order not to confuse the reader, we summarize the most relevant definitions below. Moreover, some auxiliary matrices appearing in the text are defined explicitly in this appendix. Unless stated otherwise, the notation is taken from [6].

The Hubbard–Stratonovich transformation in section 2 introduces an eight-dimensional super-matrix $Q = \{Q^{dd'}_{pp'}(gg')\}$ where the indices (p, d, g) have the following meaning: components referring to advanced and retarded Green functions, respectively, are labelled by indices $p, p' = 1, 2$. Occasionally, we find it convenient to switch to an alternative notation according to $p = 1 \rightarrow p = a$ and $p = 2 \rightarrow p = r$. The indices $g, g' = \mathrm{B}$ (boson), F (fermion) govern the grading of the matrix elements: $Q(\mathrm{BB})$ and $Q(\mathrm{FF})$ contain commuting, $Q(\mathrm{BF})$ and $Q(\mathrm{FB})$ contain anticommuting variables. The introduction of a third set of indices $d, d' = 1, 2$ is necessitated by the time-reversal invariance of the orthogonal symmetry class, as was explained in section 2.

In this paper, we structure supermatrices by first introducing 4×4 blocks referring to the p-index:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}. \tag{A1}$$

On top of that, each block $Q_{pp'}$ is decomposed with respect to the d-index:

$$Q_{pp'} = \begin{pmatrix} Q^{11}_{pp'} & Q^{12}_{pp'} \\ Q^{21}_{pp'} & Q^{22}_{pp'} \end{pmatrix} \tag{A2}$$

where each $Q^{dd'}_{pp'}$ is a 2×2 supermatrix:

$$Q^{dd'}_{pp'} = \begin{pmatrix} Q^{dd'}_{pp'}(\mathrm{BB}) & Q^{dd'}_{pp'}(\mathrm{BF}) \\ Q^{dd'}_{pp'}(\mathrm{FB}) & Q^{dd'}_{pp'}(\mathrm{FF}) \end{pmatrix}. \tag{A3}$$

It should be noted that the block matrices $Q_{pp'}$ appearing in [6] are arranged differently. In that reference,

$$Q_{pp'} = \begin{pmatrix} Q_{pp'}(\mathrm{FF}) & Q_{pp'}(\mathrm{FB}) \\ Q_{pp'}(\mathrm{BF}) & Q_{pp'}(\mathrm{BB}) \end{pmatrix}$$

with

$$Q_{pp'}(gg') = \begin{pmatrix} Q^{22}_{pp'}(gg') & Q^{21}_{pp'}(gg') \\ Q^{12}_{pp'}(gg') & Q^{11}_{pp'}(gg') \end{pmatrix}.$$

Furthermore, our subscript p is superscript in [6].

Let us introduce operators $P(p, d, g)$ projecting onto a subspace specified by the indices (p, d, g), e.g., $P(a, \mathrm{B})$ denotes a projection operator onto the advanced-boson space. By the aid of these operators, the matrices Λ, τ_3 and L_g breaking the symmetry of the 'advanced/retarded'-space, the 'GOE'-space and the graded space, respectively can be defined as

$$\begin{aligned} \Lambda &= P(a) - P(r) \\ \tau_3 &= P(2) - P(1) \\ L_g &= P(\mathrm{B}) - P(\mathrm{F}). \end{aligned} \tag{A4}$$

The matrix Q fulfils the symmetry relations (2.13) where

$$K = \begin{pmatrix} \mathbb{1}_4 & \\ & -L_g \end{pmatrix} \tag{A5}$$

$$C = \begin{pmatrix} C_0 & \\ & -C_0 \end{pmatrix}, C_0 = \begin{pmatrix} & \mathbb{1}_2 \\ L_g & \end{pmatrix} \tag{A6}$$

and the transpose A^T of a supermatrix A is defined by

$$\begin{pmatrix} A(\mathrm{BB}) & A(\mathrm{BF}) \\ A(\mathrm{FB}) & A(\mathrm{FF}) \end{pmatrix}^T = \begin{pmatrix} A(\mathrm{BB})^T & A(\mathrm{FB})^T \\ -A(\mathrm{BF})^T & A(\mathrm{FF})^T \end{pmatrix}. \tag{A7}$$

The Gaussian-type integrals underlying the technique of supersymmetric functional integration are constructed as follows. Let us first define supervectors: $\psi^T = (S_p, \chi_p)$, $\psi^\dagger = (S_p^*, \chi_p^*)$ and $\Psi^T = (\psi^T, \psi^\dagger)$ with complex commuting (anticommuting) components $S_p(\chi_p)$. The GOE-index d distinguishes between $\psi(d = 1)$ and $\psi^*(d = 2)$, respectively. By the aid of these vectors, we can formulate the relevant Gaussian integrals [18] as

$$\int \mathrm{d}[\psi] \exp(-\psi^\dagger A^{11} \psi) = \mathrm{Sdet}[A^{11}]$$

$$\int \mathrm{d}[\psi] \exp(-\Psi^\dagger \Lambda A \Psi) = (\mathrm{Sdet}[(A + CA^T C^T)/2])^{1/2} \tag{A8}$$

where $\mathrm{d}[\psi] = \prod_p 2\mathrm{d}\,\mathrm{Re}\, S_p \mathrm{d}\mathrm{Im} S_p \mathrm{d}\chi_p^* \,\mathrm{d}\chi_p$ and A^{11} is a $(d = d' = 1)$-block of an arbitrary (but in the boson–boson block positive definite) supermatrix A. As for the normalization of integrals over anticommuting variables, we follow [13]:

$$\int \mathrm{d}\chi = 0 \qquad \int \chi \,\mathrm{d}\chi = 1/\sqrt{2\pi}. \tag{A9}$$

Supertrace STr and super-determinant Sdet are respectively defined by

$$\mathrm{STr} \begin{pmatrix} A(\mathrm{BB}) & A(\mathrm{BF}) \\ A(\mathrm{FB}) & A(\mathrm{FF}) \end{pmatrix} = \mathrm{tr} A(\mathrm{FF}) - \mathrm{tr} A(\mathrm{BB})$$

$$\mathrm{Sdet} \begin{pmatrix} A(\mathrm{BB}) & A(\mathrm{BF}) \\ A(\mathrm{FB}) & A(\mathrm{FF}) \end{pmatrix} = \det[A(\mathrm{FF}) - A(\mathrm{FB}) A(\mathrm{BB})^{-1} A(\mathrm{BF})] / \det[A(\mathrm{BB})] \tag{A10}$$

$$= \exp(\mathrm{STr}[\ln A]).$$

Appendix B. Nonlinear σ-model representing a random matrix ensemble

In this appendix we derive an integral representation for the two-point correlation function

$$F(\omega, \alpha) = \mathrm{tr}\left(\frac{1}{E - \omega^+/2 - H(\alpha)}\right) \mathrm{tr}\left(\frac{1}{E + \omega^+/2 - H(\alpha)}\right) \tag{B1}$$

describing the spectral properties of an ensemble of N-dimensional random matrix Hamiltonians (1.1) in the vicinity of $E = 0$, i.e. close to the centre of the Wigner semicircle.

We restrict ourselves to the consideration of the large N limit, $N \to \infty$. $H_s(H_a)$ are real symmetric (antisymmetric) Gaussian distributed matrices defined by the moments

$$\langle H_{s,\mu\nu}\rangle_H = \langle H_{a,\mu\nu}\rangle_H = 0$$
$$\langle H_{s(a),\mu\nu}H_{s(a),\mu'\nu'}\rangle_H = \frac{\lambda^2}{N}(\delta_{\mu\mu'}\delta_{\nu\nu'} + (-)\delta_{\mu\nu'}\delta_{\nu\mu'}) \qquad \mu, \nu = 1 \ldots N \quad \text{(B2)}$$

where the angular brackets denote the ensemble average. In the extreme cases of restored ($\alpha = 0$) and full broken ($\alpha = 1$) time reversal invariance, the second moments respectively read

$$\langle H_{\mu\nu}H_{\mu'\nu'}\rangle_H = \frac{\lambda^2}{N}\begin{cases}\delta_{\mu\nu'}\delta_{\nu\mu'} + \delta_{\mu\mu'}\delta_{\nu\nu'} & \alpha = 0\\ 2\delta_{\mu\nu'}\delta_{\nu\mu'} & \alpha = 1\end{cases}. \quad \text{(B3)}$$

The appearance of a second Kronecker-δ in the first equation reflects the equality of mutually time reversed transition amplitudes $\langle\mu|H|\nu\rangle$ in the case of no symmetry breaking. As α ranges from 0 to 1, it induces the crossover to the regime of broken time reversal invariance. In the latter case, correlations between time reversed amplitudes are absent.

As far as the technicalities of the σ-model are concerned, the following presentation will be self-contained but concise. For a pedagogical introduction to the nonlinear σ-model with emphasis on random matrix theory, we refer the reader to [13]. In the derivation below, we mainly follow the lines of that reference. We begin by representing the function F in terms of a functional integral. Introducing a $4N$-component vector

$$\psi = \begin{pmatrix} S_{p\mu} \\ \chi_{p\mu}\end{pmatrix} \qquad \mu = 1 \ldots N, p = 1, 2 \quad \text{(B4)}$$

F can be written as

$$F(\omega, \alpha) = \int \mathrm{d}[\psi]\mathrm{e}^{-L[\psi,\psi^\dagger]}\psi^\dagger P(a, \mathrm{B})\psi\psi^\dagger P(r, \mathrm{B})\psi \quad \text{(B5)}$$

where

$$L[\psi, \psi^\dagger] = -\mathrm{i}\psi^\dagger\Lambda^{1/2}\left[\frac{\omega^+}{2}\Lambda + H(\alpha)\right]\Lambda^{1/2}\psi. \quad \text{(B6)}$$

As will become obvious below, the time reversal symmetry expressed by the first of (B3) must explicitly be incorporated in the parametrization of the model in order to arrive at manageable expressions after ensemble averaging. As in section 2, this can be done most naturally by regarding ψ as an analogue of a quantum mechanical wavefunction. We take the symmetry into account by principally only considering *pairs* of mutually time reversed amplitudes, $\Psi^T = (\psi^T, \psi^\dagger)$. Technically, the composite vector Ψ can be introduced by rewriting the action $L[\psi, \psi^\dagger]$ in (B6) according to

$$\begin{aligned} L[\psi, \psi^\dagger] &= \tfrac{1}{2}(L[\psi, \psi^\dagger] + L[\psi, \psi^\dagger]^T) \\ &= -\frac{\mathrm{i}}{2}\Psi^\dagger\Lambda^{1/2}\left[\frac{\omega^+}{2}\Lambda + H_s - \mathrm{i}\alpha\tau_3 H_a\right]\Lambda^{1/2}\Psi =: L[\Psi]. \end{aligned} \quad \text{(B7)}$$

Replacing ψ by Ψ in (B6), we obtain

$$F(\omega, \alpha) = \tfrac{1}{4}\int \mathrm{d}[\psi]\mathrm{e}^{-L[\Psi]}\Psi^\dagger P(a, \mathrm{B})\Psi\,\Psi^\dagger P(r, \mathrm{B})\Psi. \quad \text{(B8)}$$

The average over the Gaussian distribution (B2) can easily be carried out and we arrive at

$$\langle F(\omega,\alpha)\rangle_H = \tfrac{1}{4}\int \mathrm{d}[\psi]\mathrm{e}^{-L_0[\Psi]-L_i[\Psi]}\Psi^\dagger P(a,\mathrm{B})\Psi\,\Psi^\dagger P(r,\mathrm{B})\Psi$$

$$L_0[\Psi] = -\frac{\mathrm{i}}{2}\Psi^\dagger\frac{\omega^+}{2}\Psi \tag{B9}$$

$$L_i[\Psi] = \frac{\lambda^2}{4N}\mathrm{STr}(A^2+\alpha^2(A\tau_3)^2) \qquad A = \mathrm{i}\Lambda^{1/2}\sum_\mu \Psi_\mu\Psi_\mu^\dagger\Lambda^{1/2}.$$

Without introduction of the composite vector Ψ, we were not able to express the 'interaction term' L_i in terms of the single dyadic product A. Instead the second Kronecker-δ in (B3) would lead to more complicated correlations between components of the single vector ψ rendering the averaged expression rather clumsy.

In order to decouple the 'interaction term' by means of a Hubbard–Stratonovich transformation, we find it convenient to introduce a matrix

$$\tilde{A} = \begin{pmatrix} a_+A^{11} & a_-A^{12} \\ a_-A^{21} & a_+A^{22}\end{pmatrix} = \frac{a_++a_-}{2}A + \frac{a_+-a_-}{2}\tau_3A\tau_3 \qquad a_\pm = (1\pm\alpha^2)^{1/2} \tag{B10}$$

such that $L_i = (\lambda^2/(4N))\mathrm{STr}(\tilde{A}^2)$. After substitution of the matrix $\tilde{A}$ in (B9), the Hubbard–Stratonovich transformation results in

$$\langle F(\omega,\alpha)\rangle_H = \tfrac{1}{4}\int \mathrm{d}[\psi]\mathrm{d}[Q]\exp\left[-\frac{N}{4}\mathrm{STr}(Q^2) - \frac{\lambda}{2}\Psi^\dagger\Lambda\tilde{Q}\Psi - L_0[\Psi]\right]$$
$$\times\,\Psi^\dagger P(a,\mathrm{B})\Psi\,\Psi^\dagger P(r,\mathrm{B})\Psi \tag{B11}$$
$$\tilde{Q} = \frac{a_++a_-}{2}Q + \frac{a_+-a_-}{2}\tau_3Q\tau_3.$$

The Gaussian integration over Ψ can now straightforwardly be carried out. Using (A8) and the symmetry properties of Q, (2.13), we arrive at

$$\langle F(\omega,\alpha)\rangle_H = -\frac{N^2}{4}\int \mathrm{d}[Q]\exp\left[-\frac{N}{4}\mathrm{STr}(Q^2) + \frac{N}{2}\mathrm{STr}\ln(g(Q)^{-1})\right]$$
$$\times\,\mathrm{STr}[g(Q)P(a,\mathrm{B})]\mathrm{STr}[g(Q)P(r,\mathrm{B})] \tag{B12}$$
$$g(Q)^{-1} = \lambda\left(\tilde{Q} - \mathrm{i}\frac{\omega^+}{2\lambda}\Lambda\right).$$

In order to evaluate the integral in (B12), we employ the saddle point method, i.e. we restrict the integration to the set of Q-matrices fulfilling the stationarity condition

$$\frac{\delta}{\delta Q}(\tfrac{1}{2}\mathrm{STr}(Q^2) - \mathrm{STr}\ln(g(Q)^{-1})) = 0. \tag{B13}$$

In the limit $N\to\infty$, the integral evaluated in the saddle point approximation asymptotically converges to the exact value. In solving the saddle point equation (B13) to leading order in N, the parameters ω and α can be neglected. We are interested in spectral correlations on a short scale $\omega \simeq \Delta$. Since at the centre of the semicircle $\Delta = \pi\lambda/N$, $\omega/\lambda = \mathcal{O}(1/N)$ does not affect the solution to leading order in N. A similar argument holds for the parameter α:

Below (cf (B15)), it will turn out that already small parameter values $\alpha = \mathcal{O}(N^{-x})$, $x > 0$, suffice to break the orthogonal symmetry. In the crossover region to the regime of unitary symmetry, α as well as ω can therefore be treated as small perturbations in the sense of the N-expansion. To leading order in N the saddle point equation reads

$$Q^2 = 1. \tag{B14}$$

Expanding the exponent in (B12) to lowest order in the small parameters ω, α and taking the stationarity condition (B14) into account, we finally arrive at

$$\begin{aligned}\langle F(\omega,\alpha)\rangle_H &= -\frac{\pi^2}{4\Delta^2}\int [\mathrm{d}Q]\mathrm{e}^{-F[Q]}\mathrm{STr}[QP(a,\mathrm{B})]\mathrm{STr}[QP(r,\mathrm{B})] \\ &= \frac{\pi^2}{\Delta^2}\left(1-\tfrac{1}{4}\int [\mathrm{d}Q]\mathrm{e}^{-F[Q]}\mathrm{STr}[P(\mathrm{B})(Q-\Lambda)_{11}]\mathrm{STr}[P(\mathrm{B})(Q-\Lambda)_{22}]\right) \qquad \text{(B15)}\end{aligned}$$

$$F[Q] = \mathrm{i}\frac{\pi\omega^+}{4\Delta}\mathrm{STr}(Q\Lambda) - \frac{s_2}{8}\mathrm{STr}([Q,\tau_3]^2) \qquad s_2 = N\alpha^2.$$

Let us finally comment on an alternative way of breaking the symmetry of a real random matrix ensemble. Instead of perturbing H_s by the random contribution $\mathrm{i}\alpha H_a$, we may take H_a to be fixed. This modification does not affect the structure of the generating functional, i.e. we are again led to a functional with the form (B15). Instead of $s_2 = N\alpha^2$, however, the basic symmetry breaking parameter now reads $s_3 = \mathrm{tr}(H_a H_a^T)\alpha^2/\lambda^2$. A physical interpretation of this has been given in the introduction.

Appendix C. Parametrization of the supermatrix Q

In this appendix we derive the parametrization (3.10) for the matrix Q. Let us start from the following ansatz for the coset [13] matrix $\hat{T} = \hat{T}_C\hat{T}_D$:

$$\hat{T}_j = \begin{pmatrix} \sqrt{1+t_{12}(j)t_{21}(j)} & t_{12}(j) \\ t_{21}(j) & \sqrt{1+t_{21}(j)t_{12}(j)} \end{pmatrix}. \tag{C1}$$

where

$$t_{12}(\mathrm{C}) = \begin{pmatrix} & t_{12}^{12} \\ t_{12}^{21} & \end{pmatrix} \qquad t_{12}(\mathrm{D}) = \begin{pmatrix} t_{12}^{11} & \\ & t_{12}^{22} \end{pmatrix}. \tag{C2}$$

This representation satisfies the symmetry requirements (2.16), if we impose the condition

$$t_{21}^{11} = L_g(t_{12}^{11})^\dagger,\ t_{21}^{21} = L_g(t_{12}^{12})^\dagger \tag{C3}$$

$$(t_{12}^{ab})^T = (L_g)^b t_{21}^{\bar{b}\bar{a}}(L_g)^a \tag{C4}$$

on the 2×2 sub-supermatrices appearing in (C2) [14]. We use the convention $\bar{1} = 2$ and $\bar{2} = 1$. From (C4), we know that the sub-matrices $\{t_{12}^{11}, t_{21}^{11}\}$ and $\{t_{12}^{12}, t_{21}^{21}\}$ contain all independent variables. Note that the relations (C3) are the same as those relating the matrices t_{12} and t_{21} appearing in the square-root parametrization of the unitary model [19]

to each other. This suggests to represent them analogously in terms of an 'eigenvalue' parametrization [19]

$$t_{12}^{11} = \mathrm{i}v_1(\mathrm{D})^{-1}\sin(\hat{\theta}(\mathrm{D})/2)\mathrm{e}^{\mathrm{i}\hat{\phi}(\mathrm{D})}v_2(\mathrm{D})$$
$$t_{12}^{12} = \mathrm{i}v_1(\mathrm{C})^{-1}\sin(\hat{\theta}(\mathrm{C})/2)\mathrm{e}^{\mathrm{i}\hat{\phi}(\mathrm{C})}v_2(\mathrm{C}) \tag{C5}$$

where the quantities $v_1(j)$, $v_2(j)$, $\hat{\phi}(j)$ and $\hat{\theta}(j)$ have been defined in (3.8), (3.9) and (3.6), respectively. Combining (C1), (C2) and (C5), we obtain (3.5). After a redefinition of the anticommuting variables according to

$$\xi_1(j) \longrightarrow \xi_1^*(j) \qquad j = \mathrm{C}, \mathrm{D}$$
$$\xi_2(\mathrm{D}) \longrightarrow \xi_2^*(\mathrm{D}) \tag{C6}$$

the diagonalizing matrices $V_1(j)$ take the form given in (3.7) and (3.8). In order to obtain a parametrization for the matrix Q, the product $\hat{T} = \hat{T}_\mathrm{C}\hat{T}_\mathrm{D}$ has to be substituted in (3.2). The resulting expression can be simplified considerably by a few additional variable transformations. First, we observe that

$$V_1(\mathrm{C})V_1^{-1}(\mathrm{D}) = \begin{pmatrix} \exp(-\mathrm{i}\gamma_1\tau_3) & \\ & \exp(\mathrm{i}\gamma_2\tau_3) \end{pmatrix} \tilde{V}_1(\mathrm{C}) \tag{C7}$$

where

$$\mathrm{i}\gamma_p = (\xi_p^*(\mathrm{D})\xi_p(\mathrm{C}) - \xi_p^*(\mathrm{C})\xi_p(\mathrm{D}))/2,\ p = 1, 2 \tag{C8}$$

and $\tilde{V}_1(\mathrm{C})$ is obtained from the matrix $V_1(\mathrm{C})$ by exchanging $\xi_p(\mathrm{C})$ against

$$\tilde{\xi}_p(\mathrm{C}) = \xi_p(\mathrm{C}) - \xi_p(\mathrm{D}). \tag{C9}$$

Further, the first factor on the RHS of (C7) can be absorbed into $V_2(\mathrm{C})$ by shifting the phase $\hat{\phi}(\mathrm{C})$ according to

$$\hat{\phi}(\mathrm{C}) \longrightarrow \hat{\phi}(\mathrm{C}) - \gamma_1 + \gamma_2. \tag{C10}$$

Since (C6), (C9) and (C10) either amount to exchanging or shifting of integration variables, the associated Jacobians are unity. As a consequence of these transformations, half of the D-anticommuting variables do not appear in the parametrization of Q. This leads to a significant simplification of various terms appearing in the integrand (cf (3.14)) and makes it possible to calculate the integral manually. Without the aid of the auxiliary transformations, we had to perform the integrations by the aid of a computer algebraic system.

Finally, let us specify the integration domain of the variables appearing in (3.10). This is most economically done by relating the new parametrization to the existing one [6]. The integration over anticommuting variables is defined in a formal sense without the notion of an integration domain. In comparing the parametrizations and determining the integration domains, we therefore restrict ourselves to the respective sets of commuting variables and ignore all anticommuting variables. The 4×4-blocks Q_{12} already contain all independent variables such that it suffices to compare the blocks Q_{12} defined by (3.10) and by (5.25)—model I of [6], respectively. Equating the respective 2×2 boson–boson blocks yields

$$\cosh(\theta_\mathrm{B}(\mathrm{C}))\sinh(\theta_\mathrm{B}(\mathrm{D}))\exp(-\mathrm{i}\phi_\mathrm{B}(\mathrm{D})) = \sinh(\theta_1)\cosh(\theta_2)\exp(\mathrm{i}(\phi - \chi))$$
$$\sinh(\theta_\mathrm{B}(\mathrm{C}))\cosh(\theta_\mathrm{B}(\mathrm{D}))\exp(-\mathrm{i}\phi_\mathrm{B}(\mathrm{C})) = \cosh(\theta_1)\sinh(\theta_2)\exp(\mathrm{i}(\phi + \chi)) \tag{C11}$$

where the variables appearing on the RHS, $\theta_1, \theta_2 \in \mathcal{R}^+ = (0, \infty)$ and $\phi, \chi \in (0, 2\pi)$ are the integration variables used in [6]. Hence we are led to the identifications

$$\theta_B(C) = \theta_2 \qquad \theta_B(D) = \theta_1$$
$$\phi_B(C) = -\chi - \phi \qquad \phi_B(D) = \chi - \phi \tag{C12}$$

implying $\theta_B(j) \in \mathcal{R}^+$ and $\phi_B(j) \in (0, 2\pi)$. Comparing the fermion–fermion block we obtain

$$\cos(\theta_F(C)) \sin(\theta_F(D)) = \sin(\theta)\sqrt{t} \tag{C13}$$
$$\sin(\theta_F(C)) = \sin(\theta)\sqrt{1-t} \tag{C14}$$

and

$$\phi_F(C) = -\beta \qquad \phi_F(D) = -\alpha \tag{C15}$$

where $\theta \in (0, \pi)$ and we have reparametrized the SU(2) matrix F_1 in (5.30) of [6] by

$$F_1 = \begin{pmatrix} \sqrt{t}\exp(i\alpha) & -\sqrt{1-t}\exp(i\beta) \\ \sqrt{1-t}\exp(-i\beta) & \sqrt{t}\exp(-i\alpha) \end{pmatrix} \tag{C16}$$

with $t \in (0, 1)$ and $\alpha, \beta \in (0, 2\pi)$. Equation (C15) implies $\phi_F(C), \phi_F(D) \in (0, 2\pi)$. From (C13) and (C14) we derive $\tan(\theta_F(D)) = \sqrt{t}\tan(\theta)$, i.e. $\theta_F(D) \in (0, \pi)$. Finally, (C13) enforces $\cos(\theta_F(C)) > 0$, i.e. $\theta_F(C) \in (0, \pi/2)$. The integration intervals derived here are summarized in (3.16).

Appendix D. Calculation of the measure $[dQ]$

In this appendix, we calculate the volume element of the saddle point manifold associated with the new parametrization of the Q-matrices. Although the derivation essentially amounts to a repeated application of the chain rule, the variety of involved integration variables renders it to the most cumbersome part of the calculation. A particular detailed and pedagogical derivation of the volume element corresponding to the square-root parametrization of Q-matrices of orthogonal symmetry can be found in appendix K of [13]. Below we will follow the lines of that reference.

Our ultimate aim is to calculate the quantity

$$[dQ] = \left|\mathrm{Sdet}\left[\frac{(\delta\hat{T}')_{12}^{12}, (\delta\hat{T}')_{21}^{21}, (\delta\hat{T}')_{12}^{11}, (\delta\hat{T}')_{21}^{11}}{\delta q(C), \delta q(D)}\right]\right|^{-1} dq(C)\, dq(D) \tag{D1}$$

where $\delta\hat{T}' = (\delta\hat{T})\hat{T}^{-1}$ and $q(j) = \{\lambda_g(j), \phi_g(j), \xi_p(j), \xi_p^*(j); g = B, F; p = 1, 2\}$, $j = C, D$ is an abbreviation for all independent integration variables appearing in (3.10). We use the convention

$$\mathrm{Sdet}\left[\frac{\delta A_1, \delta A_2, \ldots}{\delta B_1, \delta B_2, \ldots}\right] := \mathrm{Sdet}\begin{bmatrix} \delta A_1/\delta B_1 & \delta A_1/\delta B_2 & \ldots \\ \delta A_2/\delta B_1 & \delta A_2/\delta B_2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}. \tag{D2}$$

The decomposition $\hat{T} = \hat{T}(\mathrm{C})\hat{T}(\mathrm{D})$ implies

$$\delta\hat{T}' = \delta\hat{T}'(\mathrm{C}) + \hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1}. \tag{D3}$$

As is obvious from the block structure (C2), the first term $\delta\hat{T}'(\mathrm{C})$, and thereby $\delta q(\mathrm{C})$, do not contribute to $(\delta\hat{T}')^{11}_{12}$ and $(\delta\hat{T}')^{11}_{21}$. This leads to a factorization of the determinant:

$$\begin{aligned}
&\mathrm{Sdet}\left[\frac{(\delta\hat{T}')^{12}_{12}, (\delta\hat{T}')^{21}_{21}, (\delta\hat{T}')^{11}_{12}, (\delta\hat{T}')^{11}_{21}}{\delta q(\mathrm{C}), \delta q(\mathrm{D})}\right] = S_1 S_2 \\
&S_1 = \mathrm{Sdet}\left[\frac{\delta\hat{T}'(\mathrm{C})^{12}_{12}, \delta\hat{T}'(\mathrm{C})^{21}_{21}}{\delta q(\mathrm{C})}\right] \\
&S_2 = \mathrm{Sdet}\left[\frac{(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{12}, (\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{21}}{\delta q(\mathrm{D})}\right].
\end{aligned} \tag{D4}$$

Let us consider each factor separately beginning with S_2. Insertion of the 'eigenvalue'-representation of the C-variables, (C5), yields

$$\begin{aligned}
(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{12} &= v_1(\mathrm{C})^{-1}\delta A^{11}_{12}(v_2(\mathrm{C})^T)^{-1} \\
(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{21} &= v_2(\mathrm{C})^T\delta A^{11}_{21}v_1(\mathrm{C})
\end{aligned} \tag{D5}$$

$$\begin{aligned}
\delta A^{11}_{12} &= \cos\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\delta B^{11}_{12}\cos\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right) + \sin\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\mathrm{e}^{\mathrm{i}\hat{\phi}(\mathrm{C})}L_g(\delta B^{11}_{12})^T\sin\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\mathrm{e}^{-\mathrm{i}\hat{\phi}(\mathrm{C})} \\
\delta A^{11}_{21} &= \cos\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\delta B^{11}_{21}\cos\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right) + \sin\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\mathrm{e}^{\mathrm{i}\hat{\phi}(\mathrm{C})}(\delta B^{11}_{21})^T L_g\sin\left(\frac{\hat{\theta}(\mathrm{C})}{2}\right)\mathrm{e}^{-\mathrm{i}\hat{\phi}(\mathrm{C})}
\end{aligned} \tag{D6}$$

$$\begin{aligned}
\delta B^{11}_{12} &= v_1(\mathrm{C})\delta\hat{T}'(\mathrm{D})^{11}_{12}v_2(\mathrm{C})^T \\
\delta B^{11}_{21} &= (v_2(\mathrm{C})^T)^{-1}\delta\hat{T}'(\mathrm{D})^{11}_{21}v_1(\mathrm{C})^{-1}
\end{aligned} \tag{D7}$$

where we have introduced auxiliary variables, δA and δB. Correspondingly we decompose the determinant S_2 into four factors, $S_2 = S_{21}S_{22}S_{23}S_{24}$, where

$$S_{21} = \mathrm{Sdet}\left[\frac{(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{12}, (\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{21}}{\delta A^{11}_{12}, \delta A^{11}_{21}}\right] \tag{D8}$$

$$S_{22} = \mathrm{Sdet}\left[\frac{\delta A^{11}_{12}, \delta A^{11}_{21}}{\delta B^{11}_{12}, \delta B^{11}_{21}}\right] \tag{D9}$$

$$S_{23} = \mathrm{Sdet}\left[\frac{\delta B^{11}_{12}, \delta B^{11}_{21}}{\delta\hat{T}'(\mathrm{D})^{11}_{12}, \delta\hat{T}'(\mathrm{D})^{11}_{21}}\right] \tag{D10}$$

$$S_{24} = \mathrm{Sdet}\left[\frac{\delta\hat{T}'(\mathrm{D})^{11}_{12}, \delta\hat{T}'(\mathrm{D})^{11}_{21}}{\delta q(\mathrm{D})}\right]. \tag{D11}$$

Two of them, S_{21} and S_{23}, turn out to be unity. This can be deduced from

$$\begin{aligned}&\mathrm{STr}[(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{12}(\hat{T}(\mathrm{C})\delta\hat{T}'(\mathrm{D})\hat{T}(\mathrm{C})^{-1})^{11}_{21}] = \mathrm{STr}[\delta A^{11}_{12}\delta A^{11}_{21}]\\&\mathrm{STr}[\delta B^{11}_{12}\delta B^{11}_{21}] = \mathrm{STr}[\delta\hat{T}'(\mathrm{D})^{11}_{12}\delta\hat{T}'(\mathrm{D})^{11}_{21}]\end{aligned} \tag{D12}$$

and the correspondence between the Jacobians associated with matrix-variable transformations on the one hand and the supertrace over the product of the corresponding infinitesimal increment matrices, i.e. the associated 'line element' on the other hand (cf [6]). In order to determine the factor S_{22}, let us write out the infinitesimal transformation (D6) explicitly in terms of matrix elements. Equation (D6) does not mix commuting and anticommuting variables whence we obtain two independent sets of equations:

$$\begin{aligned}&\begin{pmatrix}\delta A^{11}_{12}(\mathrm{BB})\\ \delta A^{11}_{12}(\mathrm{FF})\end{pmatrix} = M_{\mathrm{c}}\begin{pmatrix}\delta B^{11}_{12}(\mathrm{BB})\\ \delta B^{11}_{12}(\mathrm{FF})\end{pmatrix} \qquad \begin{pmatrix}\delta A^{11}_{21}(\mathrm{BB})\\ \delta A^{11}_{21}(\mathrm{FF})\end{pmatrix} = M_{\mathrm{c}}\begin{pmatrix}\delta B^{11}_{21}(\mathrm{BB})\\ \delta B^{11}_{21}(\mathrm{FF})\end{pmatrix}\\&\begin{pmatrix}\delta A^{11}_{12}(\mathrm{BF})\\ \delta A^{11}_{12}(\mathrm{FB})\end{pmatrix} = M_{\mathrm{a}}\begin{pmatrix}\delta B^{11}_{12}(\mathrm{BF})\\ \delta B^{11}_{12}(\mathrm{FB})\end{pmatrix} \qquad \begin{pmatrix}\delta A^{11}_{21}(\mathrm{BF})\\ \delta A^{11}_{21}(\mathrm{FB})\end{pmatrix} = \tau_3 M_{\mathrm{a}}\tau_3\begin{pmatrix}\delta B^{11}_{21}(\mathrm{BF})\\ \delta B^{11}_{21}(\mathrm{FB})\end{pmatrix}\\&M_{\mathrm{c}} = \begin{pmatrix}1 & \\ & \cos(\theta_{\mathrm{F}}(\mathrm{C}))\end{pmatrix}\\&M_{\mathrm{a}} = \begin{pmatrix}\cosh(\theta_{\mathrm{B}}(\mathrm{C})/2)\cos(\theta_{\mathrm{F}}(\mathrm{C})/2) & \mathrm{i}\sinh(\theta_{\mathrm{B}}(\mathrm{C})/2)\sin(\theta_{\mathrm{F}}(\mathrm{C})/2)\mathrm{e}^{-\mathrm{i}\epsilon}\\ \mathrm{i}\sinh(\theta_{\mathrm{B}}(\mathrm{C})/2)\sin(\theta_{\mathrm{F}}(\mathrm{C})/2)\mathrm{e}^{-\mathrm{i}\epsilon} & \cosh(\theta_{\mathrm{B}}(\mathrm{C})/2)\cos(\theta_{\mathrm{F}}(\mathrm{C})/2)\end{pmatrix}\end{aligned} \tag{D13}$$

where $\epsilon = \phi_{\mathrm{B}}(\mathrm{C}) - \phi_{\mathrm{F}}(\mathrm{C})$. The super-determinant S_{22} is just the quotient of the determinants associated with the transformations above and we obtain

$$S_{22} = \left(\frac{\mathrm{det}M_{\mathrm{a}}}{\mathrm{det}M_{\mathrm{c}}}\right)^2 = \left(\frac{\lambda_{\mathrm{B}}(\mathrm{C}) + \lambda_{\mathrm{F}}(\mathrm{C})}{2\lambda_{\mathrm{F}}(\mathrm{C})}\right)^2. \tag{D14}$$

The last factor S_{24} is the Jacobian corresponding to the transformation from $\hat{T}(\mathrm{D})$ to the actual integration variables $q(\mathrm{D})$ contained in the matrices t^{11}_{12} and t^{11}_{21}. The latter possess unitary symmetry and $\hat{T}(\mathrm{D})$ is the standard unitary coset matrix expressed in the square-root parametrization (cf (C1)). Hence we conclude that S_{24} is just the GUE volume element (cf (5.34) of [6] or (3.12, 3.13) of [19])

$$S_{24} = (\lambda_{\mathrm{B}}(\mathrm{D}) - \lambda_{\mathrm{F}}(\mathrm{D}))^2. \tag{D15}$$

The remaining factor S_1 in (D4) can be treated similarly as S_{24} by noting that the matrices $\hat{T}(\mathrm{C})$ and $\hat{T}(\mathrm{D})$ defined by (3.5) are related to each other by

$$\hat{T}(\mathrm{C}) = \begin{pmatrix}\mathbb{1}_4 & \\ & C_0^T\end{pmatrix}\hat{T}(\mathrm{D})|_{q(\mathrm{D})=q(\mathrm{C})}\begin{pmatrix}\mathbb{1}_4 & \\ & C_0\end{pmatrix} \tag{D16}$$

which gives

$$\begin{aligned}&\delta\hat{T}'(\mathrm{C})^{12}_{12} = \delta\hat{T}'(\mathrm{D})^{11}_{12}|_{q(\mathrm{D})=q(\mathrm{C})}\\&\delta\hat{T}'(\mathrm{C})^{21}_{21} = \delta\hat{T}'(\mathrm{D})^{11}_{21}|_{q(\mathrm{D})=q(\mathrm{C})}.\end{aligned} \tag{D17}$$

These equations show that S_1 equals $S_{24}|_{q(\mathrm{D})=q(\mathrm{C})}$. In other words, S_1 is the GUE volume element corresponding to the C-variables:

$$S_1 = (\lambda_{\mathrm{B}}(\mathrm{C}) - \lambda_{\mathrm{F}}(\mathrm{C}))^2. \tag{D18}$$

Combining (D1), (D4), (D14), (D15) and (D18), we finally arrive at (3.15).

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